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Bifurcation Analysis of Eigenstructure Assignment Control in a Simple Nonlinear Aircraft Model

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Aircraft systems are highly nonlinear and time varying. High-performance aircraft at high angles of incidence experience undesired coupling of the lateral and longitudinal variables, resulting in departure from normal controlled flight. The construction of a robust closed-loop control that extends the stable and decoupled flight envelope as far as possible is pursued. For the study of these systems, nonlinear analysis methods are needed. Previously, bifurcation techniques have been used mainly to analyze open-loop nonlinear aircraft models and to investigate control effects on dynamic behavior. Linear feedback control designs constructed by eigenstructure assignment methods at a fixed flight condition are investigated for a simple nonlinear aircraft model. Bifurcation analysis, in conjunction with linear control design methods, is shown to aid control law design for the nonlinear system.

Nomenclature

A, B, C	= linear system matrices
$G0d$	= desired decoupling vectors
I	= identity matrix
I_{xx}, I_{yy}, I_{zz}	= moments of inertia
K	= linear feedback matrix
L, M, N	= rolling, pitching, and yawing moments
$\mathcal{L}$	= set of prescribed eigenvalues
m, n	= number of states, inputs
p, q, r	= roll, pitch, and yaw rates
$\Re, \Im$	= real and complex spaces
S_i	= space of eigenvectors corresponding to λ_i
$\boldsymbol{u}, \boldsymbol{v}$	= linearized system input, reference input
V	= modal matrix of eigenvectors
V_0	= trim velocity
$\boldsymbol{x}, \boldsymbol{y}$	= linearized system state, output
Y, Z	= force in Y, Z direction
α, β	= angles of incidence (attack) and sideslip
δ	= perturbation
$\delta_E, \delta_A, \delta_R$	= elevator, aileron and rudder deflection
$\lambda_i, \boldsymbol{v}_i$	= eigenvalue and right eigenvector
$\dot{()}$	= time-differentiation

Subscript

OL	= open loop

I. Introduction

AIRCRAFT systems are inherently highly nonlinear and time varying. Robust control design techniques are required to cope with these characteristics and with the model uncertainty. Of particular interest is the control of high-performance aircraft at high angles of incidence, where inertial, kinematic, and aerodynamic nonlinearities are increasingly significant. Qualitative changes in the dynamic behavior of the aircraft may then occur that can induce the onset of departure from controlled flight. Motivated by the coupling of the lateral and longitudinal variables at high angles of incidence, we seek methods for constructing a robust control that optimally extends the stable and decoupled flight envelope. The aim is to find conditions and design parameters for which the best controller can be achieved for the nonlinear system.

Linearized models only represent the local dynamics of the nonlinear system. Incorporation of nonlinearities gives a more global view of the aircraft dynamics and, consequently, the insight to achieve control solutions effective over a wider flight envelope. Suitable techniques are required to investigate the effects of the nonlinearities. Bifurcation analysis is an established technique used to investigate nonlinear aircraft models.[1–6] It gives information regarding the equilibrium and periodic solutions of the aircraft system with respect to some bifurcation parameter, usually a control variable in aircraft models.

Bifurcation analysis, applied mainly to open-loop models thus far, has highlighted the effects of the nonlinearities in flight dynamics. This advance knowledge of specific dynamic behavior to be focused on gives direction to flight simulations, saving time and reducing costs.[1] Control combinations have also been investigated by using these techniques, e.g., combinations that avoid jumps when traversing the flight envelope to obtain a smooth route or combinations that utilize such jumps to increase agility. Bifurcation analysis has helped to show how particular dynamics, i.e., departure and postdeparture dynamics such as spin, are entered with regard to the aircraft states and control.[2,3] Possible recovery methods from such postdeparture dynamics can be obtained by investigating where these dynamics undergo a qualitative change in behavior and allow a possible jump back to normal flight. Such analysis of aircraft systems has related common dynamics to particular bifurcation phenomena.[4]

Some closed-loop models have also been investigated by bifurcation techniques. A simple nonlinear aircraft model using classical control methods has been examined,[5] and a high-performance aircraft model augmented by a full-authority control system has been analyzed.[6] Comparative effects of different linear design procedures on the nonlinear dynamics of an aircraft have not, however, been studied previously.

In this paper bifurcation techniques are used to analyze linear feedback controllers, constructed by eigenstructure assignment, applied to a simple nonlinear aircraft model. The aim is to incorporate the bifurcation analysis into the design procedure to investigate the effects of the control designs at equilibrium flight conditions and to improve the resulting controllers. Two eigenstructure assignment methods for controller design have been implemented. A new modified eigenstructure assignment procedure is also introduced. This technique is similar to methods proposed in Refs. 7–10, but

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is more efficient in the case where an appropriate feasible point can first be selected. The feedback gain matrices are constructed using: 1) eigenstructure assignment by a decoupling method,[9,11,12] which locally stabilizes and decouples the inputs from the outputs, reducing the undesired coupling of the lateral and longitudinal control variables; 2) robust eigenstructure assignment,[9,13–16] which reduces the sensitivity of the closed-loop system and, hence, increases the region over which the closed-loop system remains stable, but does not achieve the desired decoupling properties; and 3) the new modified eigenstructure assignment technique, which achieves the desired decoupling and at the same time reduces the sensitivity of the closed-loop system, hence, increasing the stability envelope and improving the departure characteristics of the aircraft.

The bifurcation analysis reveals that satisfactory designs for the linear systems can exhibit very different nonlinear departure characteristics and illustrates the importance of the robustness of the closed-loop system. The analysis also demonstrates that the robust decoupling controllers designed by the new modified eigenstructure assignment technique can successfully extend the stable and decoupled flight envelope and improve the nonlinear behavior of the aircraft.

In the following sections relevant concepts in bifurcation analysis are first given, followed by a description of the eigenstructure assignment control design methods for linear systems. The application of the design methods to the nonlinear system is outlined. A simple nonlinear aircraft model is then presented. Results are given, including bifurcation diagrams of the open-loop system and of the closed-loop system for various controllers designed by the eigenstructure assignment methods. The conclusions are summarized in the final section.

II. Bifurcation Analysis

Bifurcation analysis is a technique used to investigate the behavior of nonlinear systems. The technique calculates equilibrium (steady-state) and periodic (limit-cycle) solutions of the nonlinear system as a bifurcation parameter is varied, usually a control variable in the study of aircraft systems, such as elevator, aileron, or rudder deflections. A bifurcation occurs where there is a qualitative change in the system dynamics involving a change in stability of the system and often the creation or annihilation of possible solutions. At a bifurcation, an aircraft system experiences departure from its current dynamics and can jump to another equilibrium solution and a new dynamic situation. For example, at a bifurcation, the aircraft can depart from normal controlled equilibrium flight and jump into a spin, which is another steady, but undesirable, dynamic state.

The stability of the equilibrium solutions can be determined by considering the eigenvalues of the linearization of the system at the equilibrium point. A stability change, and hence a bifurcation, occurs where the eigenvalues of the linearized system cross the imaginary axis. Several types of bifurcation can occur. Possible steady-state bifurcations,[17] where a single zero eigenvalue crosses the imaginary axis, are: 1) the limit-point (turning point, saddle node) bifurcation, where a single solution branch changes stability and, beyond the critical value of the bifurcation parameter, the solution branch no longer exists; 2) the pitchfork bifurcation (in symmetric systems only, such as in some aircraft models), where at the critical parameter value the steady-state solution branch changes in stability, and two new solution branches begin; and 3) the transcritical bifurcation, where two existing solution branches of opposite stability cross, and both change in stability.

A further bifurcation of a steady-state equilibrium solution is a Hopf bifurcation. This occurs where a complex conjugate pair of eigenvalues cross the imaginary axis. The steady-state equilibrium solution changes stability, and periodic solution (limit-cycle) branches evolve from the bifurcation point.

Several software packages for bifurcation analysis exist. These use numerical continuation schemes for path following of steady-state solution branches, usually by some predictor–corrector method. An approximate initial equilibrium solution for the continuation routine to begin from is normally needed, and Newton iteration is performed if it is not accurate enough for the scheme to start from. Techniques for bifurcation detection and path following past critical points are also required in the numerical procedure. The software packages produce output files containing bifurcation information in the form of data lists, including lists of equilibrium point values, stability properties, bifurcation locations, and bifurcation types, from which bifurcation diagrams of the solution branches, displaying stability and labeled bifurcation points, can be plotted.

In this work the bifurcation analysis package AUTO94 (Ref. 18) has been used to investigate the changes in the equilibrium states of the system with respect to changes in the elevator deflection control. The other two control inputs are assumed constant at zero. Results of bifurcation data from the package are given in the form of bifurcation diagrams in Sec. VI.

III. Linear Control Design

Three eigenstructure assignment methods have been applied to the model. State feedback is assumed possible, so that all eigenvalues can be assigned exactly. First, decoupling eigenstructure assignment is applied, where the freedom in the eigenvectors is used to decouple the system modes. Second, robust eigenstructure assignment is applied, where the freedom in the eigenvectors is used to optimize the robustness, making the eigenvalues of the system as insensitive to perturbations in the system matrices as possible. Finally, a modified method is introduced that combines both of these techniques to generate closed-loop systems that are both decoupled and robust.

State Feedback Control

We consider the autonomous, linear, multivariable system with state space and output equations

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \qquad \boldsymbol{y} = C\boldsymbol{x} \tag{1}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, and $C \in \Re^{p\times n}$ are the state, input, and output matrices, respectively, and $\boldsymbol{x}$, $\boldsymbol{u}$, and $\boldsymbol{y}$ are the state, input, and output variables, respectively. Both B and C are assumed to have full rank. In the state feedback case, all state variables are available as outputs and, thus, $C = I$ (the identity matrix), giving $\boldsymbol{y} = \boldsymbol{x}$. (See the Appendix for a specific example of system matrices A and B.)

By consideration of the response equation of the system, it can be shown that both the eigenvalues and eigenvectors of the system state matrix determine the behavior of the response of the system.[19] The aim is to find a feedback (or gain) matrix K to construct a control

$$\boldsymbol{u} = K\boldsymbol{x} + \boldsymbol{v} \tag{2}$$

that gives a closed-loop system with the required eigenstructure to meet or improve specified system properties. From Eqs. (2) and (1), the closed-loop system has the form

$$\dot{\boldsymbol{x}} = (A + BK)\boldsymbol{x} + B\boldsymbol{v} \tag{3}$$

where $\boldsymbol{v}$ is the reference input.

The matrix K must be chosen to give the state matrix $A + BK$ of the closed-loop system the required set of eigenvalues. The eigenvalues are chosen to improve certain properties of the system, in particular, stability. A sufficient condition for stability is that the eigenvalues have negative real part.[19]

A multi-input system ($n \geq m \geq 2$) is assumed, giving some freedom in the choice of eigenvectors to obtain the desired eigenvalues and making it possible to improve other properties of the system. Conditions for the existence of solutions to the state feedback eigenvalue assignment problem are well known.[19] The full objective, to assign the best suitable eigenstructure, can be formulated as follows: given $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, and a set $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \Im^n$, closed under complex conjugation, find matrix $K \in \Re^{m\times n}$ and nonsingular matrix $V \in \Re^{n\times n}$ such that

$$(A + BK)V = V\Lambda \tag{4}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ and V is the modal matrix of right eigenvectors of $A + BK$.

Solutions to this problem have been established.[13] The feedback K can be calculated and is dependent on the modal matrix V of right eigenvectors assigned, i.e., $K = K(V)$. A condition that the right eigenvectors must satisfy to assign the corresponding desired eigenvalues is known.[11,13] If $\boldsymbol{v}_i$ denotes the ith column of V, the right eigenvector corresponding to λ_i, then the vector $\boldsymbol{v}_i$ must lie in a subspace S_i of dimension m; that is,

$$\boldsymbol{v}_i \in S_i \equiv S(A, B, \lambda_i) \tag{5}$$

The matrix K is calculated using a suitably constructed V satisfying this condition. As already stated, where $m > 1$, there is some freedom in the choice of the vector $\boldsymbol{v}_i$, allowing some other possible design requirements to be met.

Eigenstructure Assignment for Decoupling

The eigenvector freedom is used to achieve a specified set of desired eigenvectors to shape the response of the system, specifically to improve the flight handling qualities of the aircraft. The vectors can be chosen to decouple the system inputs from the outputs via its modes.[8,9,20] In this work decoupling the outputs (states) from specific modes only is attempted. In general, complete specification of the desired eigenvectors is neither known nor required. The designers are normally only interested in certain elements, and the remaining components are left unspecified.

Again K must be found to satisfy Eq. (4). A suitable V is constructed from vectors in the correct subspaces as close as possible, in the least squares sense, to the desired vectors by taking their projection into the required subspaces. A set of eigenvalues $\mathcal{L}$ and a set of corresponding decoupling vectors $G0d$ to be attained are chosen by the designer to achieve stability and modal decoupling. The projection is repeated once for each desired vector, generating matrix V from which the feedback matrix is constructed. Note that for complex conjugate pairs of desired eigenvalues the corresponding desired vectors must also be complex conjugate.

Robust Eigenstructure Assignment

The freedom available in the eigenvectors to be assigned is used to make the system more robust, that is, to make the assigned eigenvalues as insensitive as possible to perturbations in the system matrices. There are several different measures of robustness relating to eigenvalue sensitivity, or eigenvalue condition number.[13] With appropriate scaling $\|V^{-1}\|_F$ is used as the robustness measure, where $\|\cdot\|_F$ denotes the Frobenius norm.

A robust solution is found by selecting suitable columns $\boldsymbol{v}_i$ of V from the subspaces S_i to achieve optimal conditioning of the system. An iterative method is applied to the columns of V, attempting to make the vectors $\boldsymbol{v}_i$ as orthogonal to one another as possible, a necessary condition for optimal conditioning.[13] The stopping criterion for the iteration is that either a preset maximum number of iterations have been performed or a reduction in the robustness measure has not been achieved. Note that this does not necessarily give a global minimum of the robustness measure. If a complex conjugate pair of eigenvalues is specified, the first of the complex pair is updated as normally, and the second is taken to be the complex conjugate of the first, retaining the complex conjugacy of the eigenvectors as well as the eigenvalues. The resulting well conditioned matrix V is used to calculate K.

Combining the Two Methods

A modified method is now introduced that combines the decoupling and robustness method. In this procedure a modal matrix V is first calculated using the projection method to obtain the desired output mode decoupling. Instead of using this V to calculate the feedback K, it is used as the initial V in the robustness iteration process, with the aim of retaining the decoupling while optimizing the robustness of the closed-loop system.

The robustness iteration algorithm used[13] consists of the following steps.

1) Select an initial set of linearly independent eigenvectors $\boldsymbol{v}_i \in S_i, i = 1, 2, \ldots, n$, corresponding to the prescribed eigenvalues λ_i.

2) For $i = 1, 2, \ldots, n$, a) find $\boldsymbol{q}$ orthogonal to $\{\boldsymbol{v}_j, j \neq i\}$ and b) set $\boldsymbol{v}_i$ equal to the projection of $\boldsymbol{q}$ into the subspace S_i, normalized to unit length.

3) Repeat step 2 until the measure of robustness fails to improve by a specified tolerance.

In step 2, the selected vector $\boldsymbol{v}_i$ minimizes the sensitivity of the corresponding eigenvalue λ_i, over all vectors in S_i. Because altering one eigenvector affects the sensitivities of all of the eigenvalues of the system, the procedure must be iterated to achieve good conditioning of the entire eigenstructure.

If the initial set of eigenvectors $\{\boldsymbol{v}_i\}$ is selected to achieve the desired decoupling, then the states (or outputs) and the modes can be ordered so that the eigenvectors form two linearly independent sets

$$\left\{\boldsymbol{v}_i = \begin{pmatrix} \boldsymbol{v}_{1i} \\ \boldsymbol{0} \end{pmatrix}, \; i = 1, 2, \ldots, k\right\}$$
$$\left\{\boldsymbol{v}_i = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{v}_{2i} \end{pmatrix}, \; i = k+1, \ldots, n\right\} \tag{6}$$

In step 2 of the algorithm, if $0 \le i \le k$, then because $\boldsymbol{q}$ is orthogonal to the second set of vectors in Eq. (6), it must take the form $\boldsymbol{q}^T = (\boldsymbol{q}_1^T \;\; \boldsymbol{0})$. Similarly, if $k+1 \le i \le n$, then $\boldsymbol{q}^T = (\boldsymbol{0} \;\; \boldsymbol{q}_2^T)$. The projection $\boldsymbol{v}_i$ of $\boldsymbol{q}$ into S_i then preserves the desired decoupling. If the linear system is already decoupled, then it is always possible to select an initial set of mode decoupling eigenvectors from the required subspace for any choice of the desired eigenvalues.

For the nonlinear system, a linearization of the system at a specified equilibrium point is used to design the linear feedback controller. (The linearization is described in Sec IV.) If the design point is selected to lie on a decoupled branch of the bifurcation diagram, then the linearized system is decoupled at this point, and a feasible set to initialize the robustness iteration can always be found. The aim is then to select a robust feedback controller that extends and stabilizes the decoupled branch of the nonlinear system for the entire input parameter range.

IV. Control Design for the Nonlinear System

Application of the eigenstructure assignment design methods to the nonlinear model requires a linearization of the system about an equilibrium point. A control design is then calculated to make this particular linear approximation to the system stable and to satisfy performance requirements specified by the design. Interest lies, for a given controller, in the change in the stability, robustness, and coupling of the closed-loop equilibrium points of the nonlinear system with varying reference control.

To apply the linear design methods to a nonlinear autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{7}$$

Eq. (7) must first be linearized about a specific equilibrium point. A general equilibrium state $\boldsymbol{x}_e$ is dependent on the control parameter $\boldsymbol{u}$, that is $\boldsymbol{x}_e = \boldsymbol{x}_e(\boldsymbol{u})$. More than one equilibrium state may be associated with each control parameter. For a fixed control $\boldsymbol{u} = \boldsymbol{u}^*$, a specific equilibrium point can be written as

$$\big(\boldsymbol{x}_e(\boldsymbol{u}^*), \boldsymbol{u}^*\big)_{\mathrm{OL}} = (\boldsymbol{x}^*, \boldsymbol{u}^*)_{\mathrm{OL}} \tag{8}$$

Linearizing about this fixed equilibrium point by a Taylor's expansion and ignoring the nonlinear terms gives a linear system that approximates Eq. (7) locally to Eq. (8). If $\boldsymbol{x} = \boldsymbol{x}^* + \delta\boldsymbol{x}$ and $\boldsymbol{u} = \boldsymbol{u}^* + \delta\boldsymbol{u}$, then the linear approximation to the nonlinear system is given by

$$\delta\dot{\boldsymbol{x}} = A^*\delta\boldsymbol{x} + B^*\delta\boldsymbol{u} \tag{9}$$

where

$$A^* = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{(\boldsymbol{x}^*, \boldsymbol{u}^*)_{\mathrm{OL}}} \quad \text{and} \quad B^* = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{(\boldsymbol{x}^*, \boldsymbol{u}^*)_{\mathrm{OL}}} \tag{10}$$

The output equation is

$$\delta\boldsymbol{y} = C\delta\boldsymbol{x} = \delta\boldsymbol{x} \tag{11}$$

A control design method can now be applied to Eq. (9). A feedback matrix K^* is calculated such that the closed-loop state matrix $A^* + B^*K^*$ of the linear approximation satisfies the design requirements. The closed-loop control $\boldsymbol{u} = \boldsymbol{u}^* + \delta\boldsymbol{u} = K^*\boldsymbol{x} + \boldsymbol{v}$ is applied to the nonlinear system, giving

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, K^*\boldsymbol{x} + \boldsymbol{v}) = \boldsymbol{f}_{K^*}(\boldsymbol{x}, \boldsymbol{v}) \tag{12}$$

Bifurcation techniques are applied to Eq. (12) to investigate how well this particular design controls the nonlinear system.

V. Aircraft Model

The aircraft model used in these studies is a simple autonomous, fifth-order nonlinear system using flight data to model a twin-engined, jet fighter aircraft at a fixed flight condition in straight and level flight at a height of 1065 m, a Mach number of 0.9, a trim velocity $V_0 = 265\ \mathrm{ms}^{-1}$, a zero flight-path angle, and an angle of incidence of 2.6 deg, giving constant stability derivatives corresponding to one flight condition. The data are found in Ref. 21.

The longitudinal equations are

$$\dot{\alpha} = q - (r\sin\alpha + p\cos\alpha)\tan\beta + \frac{\cos\alpha\left(Z_\alpha\alpha + Z_{\delta_E}\delta_E\right)}{V_0\cos\beta}$$

$$\dot{q} = \left[\frac{I_{zz} - I_{xx}}{I_{yy}}\right]pr + M_\alpha\alpha + M_{\dot{\alpha}}\dot{\alpha} + M_q q + M_{\delta_E}\delta_E \tag{13}$$

and the lateral equations are

$$\dot{\beta} = p\sin\alpha - r\cos\alpha + \frac{\cos\beta\left(Y_\beta\beta + Y_{\delta_A}\delta_A + Y_{\delta_R}\delta_R\right)}{V_0}$$

$$\dot{p} = \left[\frac{I_{yy} - I_{zz}}{I_{xx}}\right]qr + L_\beta\beta + L_p p + L_r r + L_{\delta_A}\delta_A + L_{\delta_R}\delta_R \tag{14}$$

$$\dot{r} = \left[\frac{I_{xx} - I_{yy}}{I_{zz}}\right]pq + N_\beta\beta + N_p p + N_r r + N_{\delta_A}\delta_A + N_{\delta_R}\delta_R$$

Equations (13) and (14) together take the nonlinear form (7), where $\boldsymbol{x} = (\alpha, q, \beta, p, r)^T$ and $\boldsymbol{u} = (\delta_E, \delta_A, \delta_R)^T$.

A full aircraft model would have available a set of discretized aerodynamic data at many flight conditions, hence modeling the aerodynamic and kinematic nonlinearities due to the flowfield around the aircraft. In this model only inertial and kinematic nonlinearities, found in the equations of motion, are represented. Thrust and gravitational effects have been neglected. The stability derivatives are determined by linearization about the specified equilibrium flight conditions and, thus, the nonlinear model applies only locally to these conditions. Hence, this model is, in fact, not truly valid at high angles of attack. Despite the simplicity of this model, it is considered suitable for this study inasmuch as in the open loop it exhibits a bifurcation characteristic similar to more realistic aircraft models such as the high-incidence research model (HIRM).[2] A simplified model allows a greater insight into the design procedure, with the aim of eventually extending the techniques to more complicated and realistic systems.

VI. Results

Open-Loop

The nonlinear model is first run in AUTO94 (Ref. 18) with no feedback control applied, and the equilibrium state variables are found over the elevator deflection range ±29 deg. The aileron and rudder control surface deflections are constantly zero. The five plots in Fig. 1 correspond to the five state variables of the model; thus, branches shown are projections of the five-dimensional equilibrium state. Therefore, the crossing of two separate branches does not necessarily imply coinciding equilibrium states at that particular elevator value.

In the open-loop case, two separate branches are found. The straight branch corresponds to decoupled longitudinal and lateral state variables. This branch eventually becomes unstable at the Hopf bifurcation, where periodic solution branches begin (not shown in these figures). The second branch corresponds to coupled lateral and longitudinal variables, and a change in stability occurs at a limit point.

Figure 1 shows that, in the open loop, a decoupled branch already exists; however, it becomes unstable at the critical elevator deflection of approximately 5 deg. Beyond this elevator deflection the decoupled equilibrium solution becomes unstable, and the system if perturbed could jump to the stable but coupled equilibrium solution shown to exist along the second solution branch. Thus, the aim is to calculate a robust and decoupling control design that will extend the stable and decoupled branch to the complete elevator deflection range and move away or remove stable coupled branches to which the system could jump if perturbed. Note that results from such a simplified model cannot be used to make any realistic quantitative conclusions about the system dynamics.

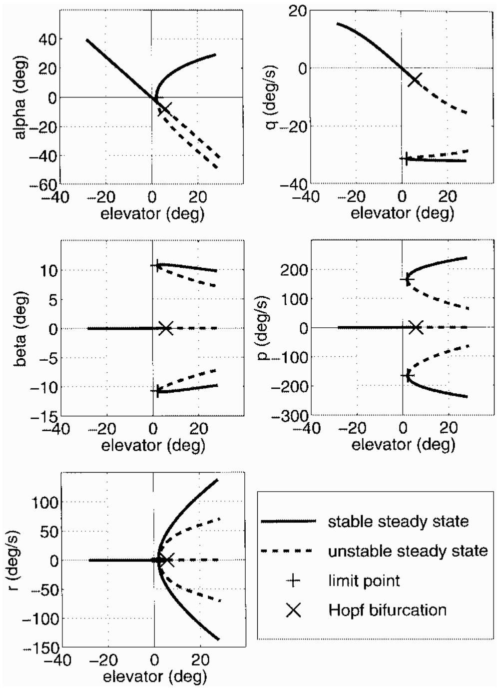

Fig. 1 Open-loop bifurcation responses.

Closed-Loop Designs

In all of the following control design examples the feedbacks are designed for a linearization of the nonlinear system that is decoupled and stable. This linearization is made at the equilibrium point (given in degrees)

$$(\boldsymbol{x}^*, \boldsymbol{u}^*)_{\mathrm{OL}} = \left(\begin{pmatrix}37.65\\15.08\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-26.90\\0\\0\end{pmatrix}\right) \tag{15}$$

and the open-loop eigenvalues are reassigned at this point. The set of assigned eigenvalues is given to four decimal accuracy by

$$\mathcal{L} = \{-0.4909 \pm 2.7996i, -0.5580 \pm 3.9212i, -0.6699\} \tag{16}$$

The system matrices of the linearized open-loop model equations at the equilibrium point (15) are displayed in the Appendix.

In the closed-loop system the bifurcation parameter is no longer the elevator control surface deflection, but is the reference input that corresponds to the pilot's longitudinal control input. The relationship between these variables must be considered when evaluating the effectiveness of a feedback control design. In our nonlinear model $\boldsymbol{u}$ represents the three control surface deflections. In the open loop these values are input directly. In the closed loop $\boldsymbol{u} = K\boldsymbol{x} + \boldsymbol{v}$, and the control surface deflections depend on the reference input $\boldsymbol{v}$, the structure of the feedback K, and the value of $\boldsymbol{x}$. Therefore, in the closed-loop system, careful monitoring of the control surface deflection values should be made when considering the validity of a control design at a particular reference input control value. Control surfaces are limited to deflection angles within a certain range due to aircraft configuration and lack control effectiveness beyond particular angles.

As in the open-loop bifurcation diagrams, the periodic solution branches are not shown and are left for investigation in future work.

Decoupling Method

Two feedback designs using the decoupling eigenstructure assignment method are generated. The only difference in the design parameters is the choice of coupling vectors $\boldsymbol{G}0d$. The choice of these vectors is made purely to decouple the longitudinal and lateral variables.

Design 1

In this design the desired decoupling vectors to be projected into the achievable subspaces are

$$G0d = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x \\ 0 & 0 & 1 & 1 & x \\ 0 & 0 & x & x & 1 \end{pmatrix} \tag{17}$$

where the x are unspecified elements. The 1s and 0s represent desired coupling and decoupling, respectively. The first two columns of $G0d$ represent the desired longitudinal modal vectors, coupled to the outputs α and q, and the remaining three columns represent the desired lateral-directional modal vectors, coupled to the outputs β, p, and r.

When projected, all specified elements are achieved exactly. The calculated design retains the stability and decoupling at the design point, but Fig. 2 shows that it does not achieve these design requirements for the nonlinear system over the total elevator deflection range and, thus, is not considered a good design.

It can be seen that the decoupled branch is stable until it undergoes a pitchfork bifurcation, where two further unstable solution branches begin. In the sub plots of the two longitudinal variables α and q in Fig. 2, the two branches are identical and plotted over one another. The decoupled branch becomes unstable after the bifurcation point. The coupled branches evolving from the bifurcation point undergo a limit-point bifurcation, where they briefly become stable, and then a Hopf bifurcation, where they become unstable again.

If this control were applied to an aircraft for elevator deflection values greater than approximately −8 deg, the aircraft is likely to jump either into one of the stable sections of the two coupled branches or into some other dynamic behavior not predicted by these results, indicating that this is not a good design. This shows the care needed when choosing decoupling vectors.

Design 2

For this design the desired mode output coupling vectors are

$$G0d = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{18}$$

where all of the zero elements specified are achieved exactly, but the 1s specified are not. Figure 3 shows that this design decouples and stabilizes the system over the whole control parameter range, indicating a good design. A second branch is also shown, which is coupled and unstable with a Hopf and limit-point bifurcation. This is one of a pair of coupled branches symmetrically located about the uncoupled branch. (Only one of the branches is shown in Fig. 3.)

Note that the set of vectors (18) is not unique in achieving a good design. Repeating the same design method as design 2 with the set of vectors

$$G0d = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & 1 \end{pmatrix} \tag{19}$$

also results in a feedback design that decouples and stabilizes the nonlinear system over the entire parameter range.

At the specified equilibrium point all of the linear closed-loop designs generated by the decoupling procedure are satisfactory, achieving exactly the specified eigenvalues and the desired decoupling. The corresponding nonlinear closed-loop systems display significantly different bifurcation behavior, however, over the whole reference input parameter range. Some designs successfully extend the stable flight envelope, whereas others are even less satisfactory than the original open-loop system. Applying the bifurcation analysis as part of the design procedure, thus, enables better control designs to be selected.

Robustness Method

Design 3

In Fig. 4, the results are given in the case where a robust feedback design is applied to the nonlinear system. The closed-loop system

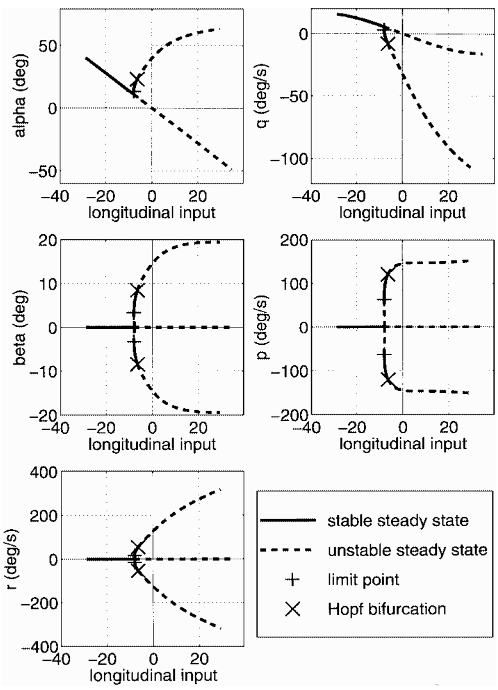

Fig. 2 Design 1, decoupling method.

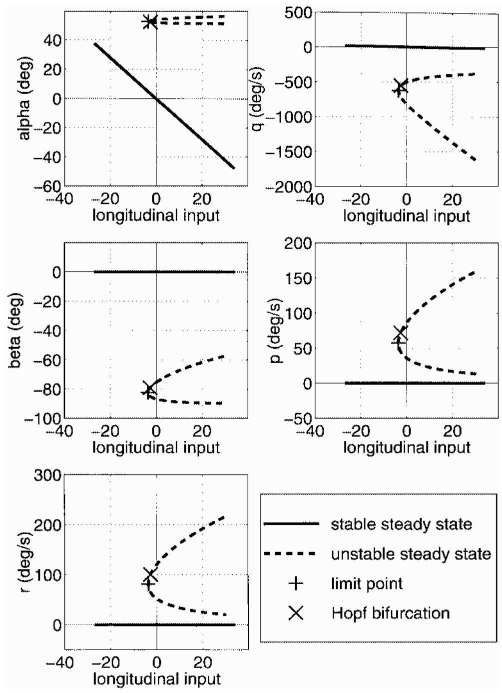

Fig. 3 Design 2, decoupling method.

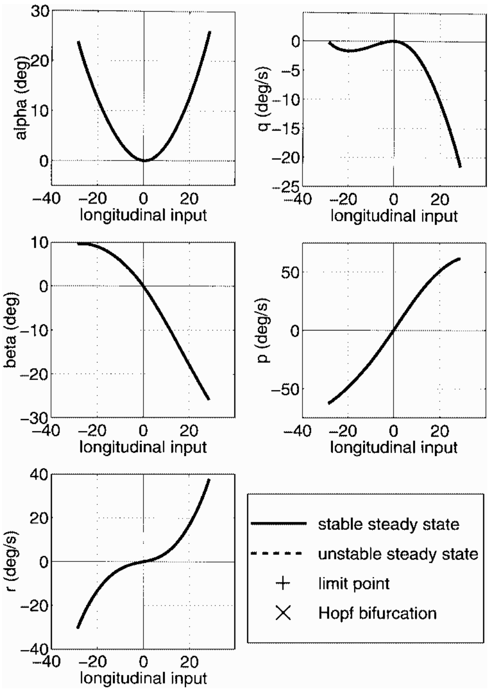

Fig. 4 Design 3, robustness method.

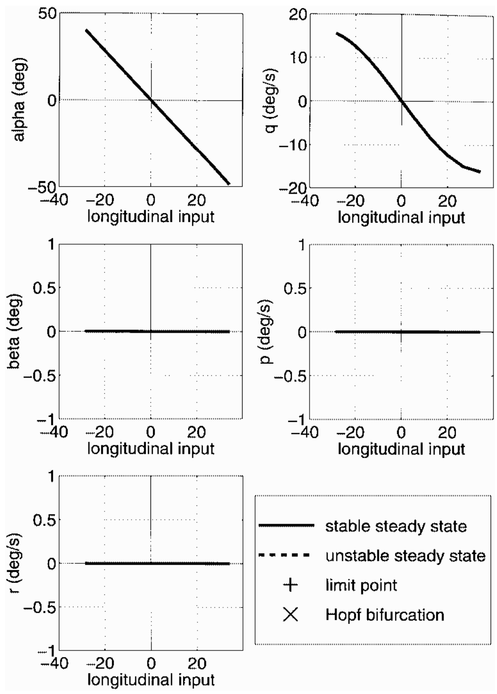

Fig. 5 Design 4, decoupling and robustness method.

remains stable over the entire bifurcation parameter range, and no bifurcation points occur; however, the system is entirely coupled in this region, which is expected because the control is not designed to decouple the system. Thus, if this design were applied to an aircraft system, it is unlikely that the system would depart to some undesired dynamic regime, but the longitudinal and lateral variables would be coupled.

Decoupling and Robustness Method

The modified technique, as described in Sec. III, is now applied to obtain a robust decoupled design. The two modal matrices calculated by the decoupling method are used as initial starting matrices in the robustness iteration. The feedback K is then constructed from the resulting V.

Design 4

Starting from the matrix V calculated in the bad feedback design, design 1, the new V found by the iteration method results in a feedback control that decouples and stabilizes the nonlinear model over the whole reference input range. The matrices V and, hence, K calculated are not equal to the matrices V and K found in the good design, design 2, using the projection method only. Bifurcation diagrams of the system states are shown in Fig. 5. No coupled branches were found in the given input parameter range, indicating that departure to another stable equilibrium state is not likely to occur with this control law.

Design 5

The same procedure is repeated on the matrix V found in the good feedback design, design 2, using the projection method only. This results in matrices V and K not equal to but containing entries close in value to those of the matrices V and K found in design 4. The resulting bifurcation diagrams are also very similar, and again the feedback gives a stable and decoupled system over the whole reference input range. The bifurcation diagram for this design is virtually identical to the diagram for design 4 (see Fig. 5).

Additional experiments have shown that the new modified eigenstructure assignment method gives similar satisfactory designs if applied at various points on the decoupled open-loop equilibrium branch, regardless of whether the corresponding linearized system is stable or unstable at the selected equilibrium point. In all cases the calculated robust feedback successfully stabilizes and decouples the system over the full reference input range.

Robustness of the feedback design at the specified equilibrium point ensures that the linearized closed-loop system is insensitive to perturbations arising, in particular, from neglected nonlinearities. The corresponding nonlinear closed-loop system is, therefore, expected to retain the assigned properties over a wider range of the reference input parameter than less robust designs. The results presented confirm that the robust designs successfully extend the stable and decoupled flight envelope and improve the departure characteristics of the nonlinear system, thus demonstrating the importance of robustness for nonlinear as well as linear systems.

VII. Conclusions

The effects of linear feedback controllers on the nonlinear dynamics of a simple aircraft model are investigated here using bifurcation techniques. The controllers are synthesized by various eigenstructure assignment procedures applied to a linearization of the aircraft model at an equilibrium point of the nonlinear system. A new modified technique is introduced that enables the design of closed-loop systems that are both robust and decoupled.

The bifurcation analysis reveals that feedback control designs that all appear satisfactory for the linearized system can exhibit very different nonlinear departure characteristics. The analysis also demonstrates that robust designs that minimize the eigenvalue sensitivity of the closed-loop system can improve the effective parameter range of the control laws acting on the nonlinear aircraft. The robust decoupling controllers designed by the new eigenstructure assignment technique are shown by the analysis to extend the stable and decoupled flight envelope successfully and to improve the nonlinear dynamical behavior of the aircraft. Thus, bifurcation analysis aids the design process, resulting in more effective control laws for the nonlinear aircraft over the entire parameter range.

Appendix: Linearized System Data

The system matrices of the fifth-order aircraft model (13) and (14) linearized about the specified open-loop equilibrium point (15) are given to four figures accuracy by

$$A^* = \begin{pmatrix} -0.2297 & 1.000 & 0 & 0 & 0 \\ -7.889 & -0.7520 & 0 & 0 & 0 \\ 0 & 0 & -0.3038 & 0.6108 & -0.7918 \\ 0 & 0 & -18.77 & -1.237 & 0.2307 \\ 0 & 0 & 5.266 & -0.2139 & -0.2450 \end{pmatrix} \tag{A1}$$

$$B^* = \begin{pmatrix} -0.04513 & 0 & 0 \\ -11.39 & 0 & 0 \\ 0 & -0.6993e\text{-}03 & -0.4296e\text{-}02 \\ 0 & 9.010 & 1.994 \\ 0 & 0.05800 & -2.634 \end{pmatrix} \quad \text{(A2)}$$

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